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Magnetization of harmonically bound charges

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Abstract. The propagator for a harmonically bound charged particle in a constant uniform magnetic field is evaluated exactly from which the corresponding partition function and density of states are obtained. Then, the equilibrium response of an assembly of harmonically bound point charges is studied using both Boltzmann and Fermi–Dirac statistics. The results are considered in relation to the ionic lattice and to some extent to the charged nucleons.

1. Introduction

The problem of the induced magnetization on charged particles bound in a general potential well is of considerable mathematical complexity. Because of this, it is difficult to reach trustworthy conclusions, and the treatment of one exactly soluble model would be of considerable interest. Such is the case of harmonically bound charges in a constant magnetic field. We consider Boltzmann and Fermi–Dirac statistics. Although the model is of a theoretical nature, it may relate to the behaviour of metallic ions in an external magnetic field and to a lesser extent to the diamagnetic behaviour of protons in a crude shell model.

In the subsequent sections we evaluate exactly the propagator of a charged particle in a harmonic oscillator well and under the influence of a constant homogeneous magnetic field. Then, the partition function and the density of states are readily obtained from the propagator. The expressions for low fields of the susceptibility in the cases of Boltzmann and Fermi–Dirac statistics are obtained via the corresponding expressions for the free energy. Some consideration of the effect of the magnetic field on the specific heat of the metallic lattice is also given.

After completion of the present work we found that Darwin (1930)† also obtained the high temperature limit of the magnetic susceptibility of a system of harmonically bound electrons. There the harmonic well was used as a device to replace the bounding walls of a system of free electrons in a magnetic field.

2. The propagator

Perhaps the simplest and most elegant way to obtain the propagator of the Schrödinger equation for a harmonically bound charged particle in a magnetic field is by functional integration. In this approach it is more natural to deal with Lagrangians than Hamiltonians.

We shall be concerned with a charged particle in a harmonic oscillator well

$$\frac{m}{2}\Omega^2\boldsymbol{\xi}^2$$

and a constant magnetic field $\boldsymbol{B} = (0, 0, B)$. In this case the Lagrangian is

$$L = \frac{m}{2}\dot{\boldsymbol{\xi}}^2 - \frac{m}{2}\Omega^2\boldsymbol{\xi}^2 + \frac{m}{2}\omega(\xi_1\dot{\xi}_2 - \xi_2\dot{\xi}_1) \qquad (2.1)$$

where $\omega = eB/m$ is the cyclotron frequency.

† I am grateful to the referee for drawing my attention to this paper.

Let us introduce the (2×2) matrices

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.2}$$

which obey the relation

$$J^2 = -I \tag{2.2a}$$

that is, the matrix J plays the role of the '(2×2) imaginary unit'.

Let us also denote the component of $\boldsymbol{\xi}$ perpendicular to the magnetic field by $\boldsymbol{\xi}_\perp$, that is

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.$$

The Lagrangian (2.1) can be split into two independent terms, one of which is entirely dependent on the component $\boldsymbol{\xi}_\perp$ and the other on the parallel component ξ_3, that is

$$L[\boldsymbol{\xi}] = L_\perp[\boldsymbol{\xi}_\perp] + L_\parallel[\xi_3]. \tag{2.3}$$

Using (2.1) and (2.2) we have

$$L_\perp[\boldsymbol{\xi}_\perp] = \frac{m}{2}\dot{\boldsymbol{\xi}}_\perp{}^2 - \frac{m}{2}\Omega^2\boldsymbol{\xi}_\perp{}^2 - \frac{m}{2}\omega\tilde{\boldsymbol{\xi}}_\perp J\dot{\boldsymbol{\xi}}_\perp \tag{2.3a}$$

$$L_\parallel[\xi_3] = \frac{m}{2}\dot{\xi}_3{}^2 - \frac{m}{2}\Omega^2\xi_3{}^2. \tag{2.3b}$$

Now since, there is no term in the decomposition (2.3) of the Lagrangian mixing the variables $\boldsymbol{\xi}_\perp$ and ξ_3, the propagator K factorizes as follows:

$$K = K_\perp \,.\, K_\parallel \tag{2.4}$$

where $K_\parallel$ corresponds to the Lagrangian $L_\parallel$ given in (2.3b), and which is the Lagrangian for a one dimensional harmonic oscillator and is given by Feynman and Hibbs (1965) as

$$K_\parallel(x_3t|x_3'0) = \left(\frac{m\Omega}{2\pi\mathrm{i}\hbar\sin\Omega t}\right)^{1/2} \times\exp\left(\frac{\mathrm{i}}{\hbar}\frac{m\Omega}{2\sin\Omega t}\{\cos\Omega t(x_3{}^2+x_3'^2)-2x_3x_3'\}\right). \tag{2.5}$$

We now wish to evaluate the propagator $K_\perp$ for the perpendicular component $\boldsymbol{\xi}_\perp$. As is well known the propagator takes the form of a conditional path integral

$$K_\perp(\boldsymbol{x}_\perp t|\boldsymbol{x}_\perp'0) = \int\exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^t L_\perp[\boldsymbol{\xi}_\perp(\tau)]\,\mathrm{d}\tau\right\}\mathscr{D}[\boldsymbol{\xi}_\perp(\tau)] \tag{2.6}$$

$$\boldsymbol{\xi}_\perp(0) = \boldsymbol{x}_\perp'$$
$$\boldsymbol{\xi}_\perp(t) = \boldsymbol{x}_\perp$$

where the path differential is given by

$$\mathscr{D}[\boldsymbol{\xi}_\perp(\tau)] = \left(\prod_{0\leqslant\tau<t}\frac{m}{2\pi\mathrm{i}\hbar\,\mathrm{d}\tau}\right)^{2/2}\prod_{0<\tau<t}\mathrm{d}\xi_1(\tau)\,\mathrm{d}\xi_2(\tau). \tag{2.6a}$$

Since the Lagrangian $L_\perp$ given in (2.3*a*) is quadratic the path integral (2.6) can be evaluated as

$$K_\perp(\boldsymbol{x}_\perp t|\boldsymbol{x}_\perp'0) = \frac{m}{2\pi i\hbar\{\det D(t)\}^{1/2}}\exp\left(\frac{i}{\hbar}S_\perp(\boldsymbol{x}_\perp t|\boldsymbol{x}_\perp'0)\right) \tag{2.7}$$

where $S_\perp(\boldsymbol{x}_\perp t|\boldsymbol{x}_\perp'0)$ is the classical action (Hamilton's principal function) for the perpendicular component of our particle evolving via the equations of motion

$$\frac{d}{d\tau}\frac{\partial L_\perp}{\partial\dot{\boldsymbol{\xi}}_\perp} = \frac{\partial L_\perp}{\partial\boldsymbol{\xi}_\perp} \tag{2.8}$$

which, utilizing (2.3*a*) can be written explicitly as

$$\ddot{\boldsymbol{\xi}}_\perp + \omega J\dot{\boldsymbol{\xi}}_\perp + \Omega^2\boldsymbol{\xi}_\perp = \mathbf{0}. \tag{2.8a}$$

$D(t)$ appearing in (2.7) is a (2×2) matrix and, can be shown, using equations (5.8, 5.22, 5.23) from Papadopoulos (1968) with

$$g = -\frac{im}{2\hbar}I \qquad C = \frac{im}{2\hbar}\Omega^2 I \qquad \alpha = -\frac{\omega}{2}I$$

to satisfy the differential equation

$$\ddot{D} - \frac{\omega}{2}\dot{D}J + \frac{\omega}{2}D\tilde{J}D^{-1}\dot{D} + D\left\{\Omega^2 + \left(\frac{\omega}{2}\right)^2\right\} = 0 \tag{2.9}$$

under the initial conditions

$$D(0) = 0 \qquad \dot{D}(0) = I. \tag{2.9a}$$

Fortunately the required solution can be obtained in the form ϕI, ϕ being a scalar and (2.9) reduces to

$$\ddot{D} + D\Omega'^2 = 0 \tag{2.10}$$

where we have introduced

$$\Omega' = \{\Omega^2 + (\omega/2)^2\}^{1/2}. \tag{2.10a}$$

Taking account of the boundary conditions (2.9*a*) we have

$$D(\tau) = \frac{\sin\Omega'\tau}{\Omega'}I. \tag{2.10b}$$

Later on we shall extract $D(t)$ from the classical action, using a result of Van Vleck (1928) and Pauli (1951).

We turn now to the evaluation of the classical action $S_\perp$. We require the solution of the equations of motion (2.8*a*) under the end conditions

$$\boldsymbol{\xi}_\perp(0) = \boldsymbol{x}_\perp' \qquad \boldsymbol{\xi}_\perp(t) = \boldsymbol{x}_\perp. \tag{2.11}$$

One might be tempted to decouple the equations of motion (2.8*a*), but apparently it is more natural to treat them as they are. The auxiliary equation associated with (2.8*a*) is

$$R^2 + \omega JR + \Omega^2 I = 0. \tag{2.12}$$

It is satisfied by the matrices

$$R_1 = \left(-\frac{\omega}{2}+\Omega'\right)J \qquad R_2 = -\left(\frac{\omega}{2}+\Omega'\right)J. \tag{2.12a}$$

Therefore the general solution of (2.8*a*) is

$$\boldsymbol{\xi}_\perp(\tau) = \exp\left\{J\left(\Omega'-\frac{\omega}{2}\right)\tau\right\}\boldsymbol{a}+\exp\left\{-J\left(\Omega'+\frac{\omega}{2}\right)\tau\right\}\boldsymbol{b}. \tag{2.13}$$

The matrix of the form $\exp(J\phi)$ appearing in the solution (2.13) is just the plane rotation matrix, that is

$$\exp(J\phi) = \cos\phi I+\sin\phi J \tag{2.13a}$$

as can easily be deduced from the property $J^2 = -I$, given (2.2*a*). Specification of the coefficients $\boldsymbol{a}$, $\boldsymbol{b}$ via the end conditions (2.11), leads to the required classical path, which we shall denote by $\boldsymbol{X}_\perp(\tau)$. It is then a matter of routine algebra to obtain

$$\boldsymbol{X}_\perp(\tau) = \exp\left(-J\frac{\omega}{2}\tau\right)\left[\cos\Omega'\tau\,\boldsymbol{x}_\perp' + \frac{\sin\Omega'\tau}{\sin\Omega' t}\left\{\exp\left(J\frac{\omega}{2}t\right)\boldsymbol{x}_\perp - \cos\Omega' t\boldsymbol{x}_\perp'\right\}\right] \tag{2.14}$$

where for the derivation of (2.14) we have made use of (2.13*a*). In the case of homogeneous quadratic Lagrangians (which is the case here) the classical action between the points $(0, \boldsymbol{x}_\perp')$ and $(t, \boldsymbol{x}_\perp)$ becomes

$$\begin{aligned} S_\perp(\boldsymbol{x}_\perp t|\boldsymbol{x}_\perp' 0) &= \frac{m}{2}\{\boldsymbol{x}_\perp\,.\,\dot{\boldsymbol{X}}_\perp(t)-\boldsymbol{x}_\perp'\,.\,\dot{\boldsymbol{X}}_\perp(0)\} \\ &= \frac{m\Omega'}{2\sin\Omega' t}\left(\cos\Omega' t(\boldsymbol{x}_\perp{}^2+\boldsymbol{x}_\perp'^2)-2\cos\frac{\omega}{2}t(\boldsymbol{x}_\perp\,.\,\boldsymbol{x}_\perp')\right) \\ &\quad + \frac{m\Omega'}{\sin\Omega' t}\sin\frac{\omega}{2}t(\boldsymbol{x}_\perp\times\boldsymbol{x}_\perp')_3. \end{aligned} \tag{2.15}$$

Combining (2.4), (2.5), (2.7), (2.10*b*) and (2.15) we find for the required propagator the result:

$$\begin{aligned} K(\boldsymbol{x}t|\boldsymbol{x}'0) &= \frac{m\Omega'}{2\pi i\hbar\sin\Omega' t}\left(\frac{m\Omega}{2\pi i\hbar\sin\Omega t}\right)^{1/2} \\ &\times\exp\left\{\frac{i}{\hbar}\frac{m\Omega'}{2\sin\Omega' t}\left(\cos\Omega' t(\boldsymbol{x}_\perp{}^2+\boldsymbol{x}_\perp'^2)-2\cos\frac{\omega}{2}t(\boldsymbol{x}_\perp\,.\,\boldsymbol{x}_\perp')+2\sin\frac{\omega}{2}t(\boldsymbol{x}_\perp\times\boldsymbol{x}_\perp')_3\right)\right\} \\ &\times\exp\left(\frac{i}{\hbar}\frac{m\Omega}{2\sin\Omega t}\{\cos\Omega t(x_3{}^2+x_3'^2)-2x_3x_3'\}\right). \end{aligned} \tag{2.16}$$

It was pointed out earlier on that the matrix $D(t)$ appearing in (2.7) can also be extracted from the classical action. The details can be found in a beautiful paper by De Witt (1957). Here we just quote the result:

$$mD^{-1}(t) = -\frac{\partial^2 S_\perp}{\partial\boldsymbol{x}_\perp\partial\boldsymbol{x}_\perp'} \tag{2.17}$$

which easily yields the result obtained from (2.10*b*).

It is easy to verify that when $\omega \to 0$ (2.16) goes to the harmonic oscillator propagator, whereas when $\Omega \to 0$ the propagator goes to that of an electron in a magnetic field. From (2.16) it follows that the pre-exponential factor associated with the perpendicular components is affected by the magnetic field in that Ω is replaced by $\{(\Omega^2+(\omega/2)^2\}^{1/2}$. However, the influence of the magnetic field in the spatial variation of the propagator is more pronounced.

Next we shall obtain the density of states of our charge. We shall do so from the partition function, which we shall use for other purposes. It is well known that the partition function can be obtained from the propagator as

$$Z(\beta) = \int K(\boldsymbol{x}, -\mathrm{i}\hbar\beta|\boldsymbol{x}, 0)\,\mathrm{d}\boldsymbol{x}$$

$$= \left[2^3 \sinh\left\{\tfrac{1}{2}\beta\hbar\left(\Omega'+\frac{\omega}{2}\right)\right\}\sinh\left\{\tfrac{1}{2}\beta\hbar\left(\Omega'-\frac{\omega}{2}\right)\right\}\sinh(\tfrac{1}{2}\beta\hbar\Omega)\right]^{-1} \tag{2.18}$$

where for the derivation of (2.18) we have made use of (2.16) for the propagator.

The partition function (2.18) amounts to the partition function of an anisotropic oscillator with fundamental frequencies:

$$\Omega_1 = \Omega'+\frac{\omega}{2} \qquad \Omega_2 = \Omega'-\frac{\omega}{2} \qquad \Omega_3 = \Omega. \tag{2.18a}$$

The density of states is the inverse Laplace transform of $Z(\beta)$.

Now, using the relation

$$\tfrac{1}{2}\{\sinh(\tfrac{1}{2}\beta\hbar\Omega)\}^{-1} = \int_0^\infty \sum_{n=0}^{\infty} \delta\{\epsilon-(n+\tfrac{1}{2})\hbar\Omega\}\exp(-\beta\epsilon)\,\mathrm{d}\epsilon \tag{2.19}$$

for the three frequencies $\Omega'+(\omega/2)$, $\Omega'-(\omega/2)$ and Ω, and applying the convolution theorem to (2.18) we find for the density of states the result

$$g(\epsilon) = \sum_{n_1,n_2,n_3=0}^{\infty} \delta(\epsilon-\epsilon_{n_1n_2n_3}) \tag{2.20}$$

where $\epsilon_{n_1n_2n_3}$ are the various energy levels:

$$\epsilon_{n_1n_2n_3} = (n_1+\tfrac{1}{2})\hbar\left(\Omega'+\frac{\omega}{2}\right)+(n_2+\tfrac{1}{2})\hbar\left(\Omega'-\frac{\omega}{2}\right)+(n_3+\tfrac{1}{2})\hbar\Omega. \tag{2.20a}$$

Expression (2.20*a*) for the energy levels gives the Zeeman effect for harmonically bound electrons. It should be noted that the degeneracy of the various energy levels of the isotropic oscillator is generally removed upon the introduction of the magnetic field.

3. Simple applications

We shall be concerned with two cases where charged particles are more or less harmonically bound:

(i) The Einstein model for an ionic lattice under the influence of an external magnetic field. In this case we obtain the susceptibility of the ionic lattice and the effect of the magnetic field on the lattice specific heat. The susceptibility will be extremely small, since the magnetic moment of an ion is very small due to the large mass of the latter. Nevertheless this quantity is shown to be different from zero, however small it might be. Similar remarks apply to the effect of the magnetic field on the lattice specific heat.

(ii) The protons in a crude shell model (with the nucleons considered independently harmonically bound) under the influence of an external magnetic field. Here, we shall be mainly interested in the steady susceptibility at absolute zero temperature.

3.1. *The Einstein model*

According to the Einstein model the ions of the lattice vibrate about their mean positions independently of each other. Under the influence of an external magnetic field the quantal motion of each ion is described by our propagator (2.16). Now, for the lattice particles Boltzmann statistics are appropriate and since the ions are considered independent, the single-particle partition function given in (2.18) suffices to provide all the necessary thermodynamic information.

The Boltzmann free energy

$$F = -NkT \ln Z(\beta)$$

takes the form:

$$F = NkT\left[\ln \sinh\left\{\tfrac{1}{2}\beta\hbar\left(\Omega' + \frac{\omega}{2}\right)\right\} + \ln \sinh\left\{\tfrac{1}{2}\beta\hbar\left(\Omega' - \frac{\omega}{2}\right)\right\} + \ln \sin(\tfrac{1}{2}\beta\hbar\Omega) + 3\ln 2\right]. \tag{3.1}$$

The lattice energy under the influence of the magnetic field will be given by

$$\begin{aligned} E &= \frac{\partial}{\partial\beta}(\beta F) \\ &= \frac{N}{2}\left[\hbar\left(\Omega' + \frac{\omega}{2}\right)\coth\left\{\tfrac{1}{2}\beta\hbar\left(\Omega' + \frac{\omega}{2}\right)\right\} + \hbar\left(\Omega' - \frac{\omega}{2}\right)\coth\left\{\tfrac{1}{2}\beta\hbar\left(\Omega' - \frac{\omega}{2}\right)\right\}\right. \\ &\qquad \left. + \hbar\Omega\coth(\tfrac{1}{2}\beta\hbar\Omega)\right]. \end{aligned} \tag{3.2}$$

The lattice heat capacity is given by

$$\begin{aligned} C &= \frac{\partial E}{\partial T} \\ &= Nk\left(\frac{\{\hbar(\Omega' + \frac{1}{2}\omega)/2kT\}^2}{\sinh^2\{\hbar(\Omega' + \frac{1}{2}\omega)/2kT\}} + \frac{\{\hbar(\Omega' - \frac{1}{2}\omega)/2kT\}^2}{\sinh^2\{\hbar(\Omega' - \frac{1}{2}\omega)/2kT\}} + \frac{(\hbar\Omega/2kT)^2}{\sinh^2(\hbar\Omega/2kT)}\right). \end{aligned} \tag{3.3}$$

The orders of magnitude of Ω and ω are $10^{13}\ \mathrm{s}^{-1}$ and $10^{8}\ \mathrm{s}^{-1}$ (for $B \sim 10^4$ G). Under these circumstances Ω' is practically Ω. Therefore, from (3.3) it follows that the heat capacity is practically unaffected by magnetic fields up to 10^4 G and even higher.

Using (3.1) for the free energy of the ionic lattice, we obtain for the magnetization per unit volume

$$\begin{aligned} M_i &= -\frac{1}{V}\frac{\partial F}{\partial B} \\ &= -\frac{N}{V}\frac{\mu_i}{2}\left[\coth\left\{\tfrac{1}{2}\beta\hbar\left(\Omega' + \frac{\omega}{2}\right)\right\}\left(\frac{\omega}{2\Omega'} + 1\right)\right. \\ &\qquad \left. + \coth\left\{\tfrac{1}{2}\beta\hbar\left(\Omega' - \frac{\omega}{2}\right)\right\}\left(\frac{\omega}{2\Omega'} - 1\right)\right] \end{aligned} \tag{3.4}$$

where $\mu_i = e_i\hbar/2m_i$ is the magnetic moment of an ion.

The magnetic susceptibility per unit volume is then given by

$$\chi_{\mathrm{i}} = \mu_0 \frac{\partial M_{\mathrm{i}}}{\partial B}$$

$$= -\mu_0 \frac{N}{V}\frac{\mu_{\mathrm{i}}^2}{4kT}\left\{1-\left(\frac{\omega}{2\Omega'}\right)^2\right\}\left[\coth\left\{\tfrac{1}{2}\beta\hbar\left(\Omega'+\frac{\omega}{2}\right)\right\}+\coth\left\{\tfrac{1}{2}\beta\hbar\left(\Omega'-\frac{\omega}{2}\right)\right\}\right]$$

$$+\mu_0 \frac{N}{V}\frac{\mu_{\mathrm{i}}^2}{4kT}\left[\frac{\left(1+\dfrac{\omega}{2\Omega'}\right)^2}{\sinh^2\left\{\tfrac{1}{2}\beta\hbar\left(\Omega'+\dfrac{\omega}{2}\right)\right\}}+\frac{\left(1-\dfrac{\omega}{2\Omega'}\right)^2}{\sinh\left\{\tfrac{1}{2}\beta\hbar\left(\Omega'-\dfrac{\omega}{2}\right)\right\}}\right]. \tag{3.5}$$

where μ_0 is the permeability of free space.

Since, due to the orders of magnitude of Ω and ω, we are practically in the $\omega = 0$ region, let us evaluate the steady susceptibility. We have, letting $\omega \to 0$

$$\chi_{\mathrm{i}} = -\mu_0 \frac{N}{V}\frac{\mu_{\mathrm{i}}^2}{2kT}\left\{\frac{2kT}{\hbar\Omega}\coth\left(\frac{\hbar\Omega}{2kT}\right)-\frac{1}{\sinh^2(\hbar\Omega/2kT)}\right\}. \tag{3.5a}$$

Putting $\hbar\Omega/2kT = x$, it is easy to verify that the function

$$\frac{1}{x}\coth x-\frac{1}{\sinh^2 x} \qquad \text{for } x > 0$$

is positive, and therefore the ionic lattice at low fields is diamagnetic.

Using (3.2) and (3.3) at $\omega = 0$, the free field lattice energy and heat capacity are

$$E_0 = \tfrac{3}{2}N\hbar\Omega\coth\left(\frac{\hbar\Omega}{2kT}\right) \qquad C_0 = 3Nk\frac{(\hbar\Omega/2kT)^2}{\sinh^2(\hbar\Omega/2kT)}. \tag{3.6}$$

With the aid of (3.6) we can express (3.5*a*) in terms of the zero field properties of the lattice E_0 and C_0. So, the ionic susceptibility at zero field is given by

$$\chi_{\mathrm{i}} = -\mu_0\frac{2}{3V}\frac{\mu_{\mathrm{i}}^2}{\hbar\Omega}\frac{E_0-C_0T}{\hbar\Omega} \qquad \text{at } B = 0. \tag{3.7}$$

It should be noted here that classically the quantity E_0-C_0T is zero, whereas quantally it differs from zero. Therefore the existence of magnetization in the ionic lattice (however small it might be) is a purely quantum effect.

3.2. *The crude shell model*

Let us now turn to the magnetization of harmonically bound charges which obey Fermi–Dirac statistics.

As pointed out earlier on we shall restrict the discussion to the absolute zero temperature. It is well known that the free energy at $T = 0$ K is just the ground energy of the system, that is

$$F_0 = \int_0^{\zeta} \epsilon \sum_{\boldsymbol{n}} \delta(\epsilon-\epsilon_0-\boldsymbol{n}\,.\,\hbar\boldsymbol{\Omega})\,\mathrm{d}\epsilon \tag{3.8}$$

where the index $\boldsymbol{n}$ stands collectively for the three quantum numbers n_1, n_2, n_3 of the energy eigenvalues given in (2.20a). $\epsilon_0(=\hbar\Omega'+\frac{1}{2}\hbar\Omega)$ is the single oscillator ground energy and $\boldsymbol{\Omega}=(\Omega_1, \Omega_2, \Omega)$ given by (2.18a).

ζ is the Fermi level defined by

$$\sum_{\boldsymbol{n}} \theta(\zeta-\epsilon_0-\boldsymbol{n} \cdot \hbar\boldsymbol{\Omega}) = N. \tag{3.9}$$

Equation (3.9) defines ζ so that the number of lattice points (n_1, n_2, n_3) for which

$$\epsilon_{\boldsymbol{n}} \leqslant \zeta \tag{3.9a}$$

equals N, the number of particles in the system. Thus, the lattice points which satisfy (3.9) are all the lattice points inside and on the tetrahedron formed by the coordinate planes of the first octand and the plane

$$x_1\hbar\Omega_1+x_2\hbar\Omega_2+x_3\hbar\Omega = \zeta-\epsilon_0. \tag{3.9b}$$

Using (3.9) we find for the magnetization per unit volume.

$$M = -\frac{1}{V}\frac{\partial F_0}{\partial B} = -\frac{1}{V}\sum_{\boldsymbol{n}} \theta(\zeta-\epsilon_0-\boldsymbol{n} \cdot \hbar\boldsymbol{\Omega})\left(\frac{\partial\epsilon_0}{\partial B}+\boldsymbol{n} \cdot \hbar\frac{\partial\boldsymbol{\Omega}}{\partial B}\right). \tag{3.10}$$

Using (3.9), (3.10) becomes

$$M = -\frac{N}{V}\mu^2\frac{B}{\hbar\Omega'}-\frac{1}{V}\sum_{\boldsymbol{n}} \theta(\zeta-\epsilon_0-\boldsymbol{n} \cdot \hbar\boldsymbol{\Omega})(n_1\hbar\Omega_1-n_2\hbar\Omega_2)\frac{\mu}{\hbar\Omega'}. \tag{3.11}$$

The term $-N\mu^2B/V\hbar\Omega'$ represents the steady part of the magnetization, whereas the series term in (3.11) is a small oscillatory term as a function of B. The oscillations of the magnetization become smaller and smaller as the spacing of the energy levels becomes closer and closer. It is worth noticing that the steady magnetization comes solely from the lowest energy of the harmonic oscillator. Therefore if the measurement of the magnetization of harmonically bound charges is feasible, this provides a test for the energy value $\frac{3}{2}\hbar\Omega$ to be the zero point energy of the harmonic oscillator.

Finally, we show that in the limit of highly packed levels the series term becomes zero. This is done by converting the sum in (3.11) into an integral, which in this case is permissible.

We introduce

$$\epsilon_1 = n_1\hbar\Omega_1 \qquad \epsilon_2 = n_2\hbar\Omega_2 \qquad \epsilon_3 = n_3\hbar\Omega. \tag{3.12a}$$

On changing the quantum numbers n_j by unity the energy changes are

$$\Delta\epsilon_1 = \hbar\Omega_1 \qquad \Delta\epsilon_2 = \hbar\Omega_2 \qquad \Delta\epsilon_3 = \hbar\Omega. \tag{3.12b}$$

Then, the volume of a unit cell is given by

$$\Delta\epsilon_1\Delta\epsilon_2\Delta\epsilon_3 = (\hbar\Omega)^3. \tag{3.12c}$$

Therefore the series term in (3.11), in terms of the energy components takes the form

$$M_1 = -\frac{\mu}{V(\hbar\Omega')}\sum(\epsilon_1-\epsilon_2)\frac{\Delta\epsilon_1\Delta\epsilon_2\Delta\epsilon_3}{(\hbar\Omega)^3} \tag{3.13}$$

where the summation is taken over all points $(\epsilon_1, \epsilon_2, \epsilon_3)$ obtained via the transformation (3.12a) when the n_j run over all the lattice points of the tetrahedron as before.

In summing (3.13), the deviation from zero will come from the anisomery and the size of the edges of the unit cells. This deviation goes to zero as the size of the energy cells tends to zero.

In the limit of highly packed energy levels (3.13) goes to

$$M_1 = -\frac{\mu/\hbar\Omega'}{V(\hbar\Omega)^3}\int_0^\zeta d\epsilon_1 \int_0^{\zeta-\epsilon_1} d\epsilon_2 \int_0^{\zeta-\epsilon_1-\epsilon_2} d\epsilon_3(\epsilon_1-\epsilon_2) = 0 \qquad (3.13a)$$

where in (3.13a) we used for convenience ζ in place of $\zeta-\epsilon_0$.

Using (3.11) and (3.13) we have for the susceptibility

$$\chi = -\mu_0 \frac{N}{V}\frac{\mu^2}{\hbar\Omega'}\left(1-\frac{\omega}{2\Omega'}\right)+\frac{\partial M_1}{\partial B}. \qquad (3.14)$$

The first term in (3.14) is the steady susceptibility, which is diamagnetic. The term $\partial M_1/\partial B$ is bound to exhibit sharp peaks at certain values of B for which net migration between the populations of the quantum numbers 1 and 2 takes place, as a result of the Fermi–Dirac statistics.

Finally, for small fields the steady susceptibility becomes

$$\chi = -\mu_0 \frac{N}{V}\frac{\mu^2}{\hbar\Omega}. \qquad (3.15)$$

4. Conclusions

The present calculations show the existence of magnetization for harmonically bound charges as a quantum effect. They also show that the order of magnitude of the ionic magnetization is such that it cannot possibly obscure the magnetic measurements of the conduction electrons. It should of course be noted here that we are excluding electronic transitions in the ions. Similar remarks apply to the measurements of the electronic specific heat.

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